

# Algorithms for Submodular Matroid Secretary Problems Under Transversal Matroids and Partition Matroids

Bo Tang\*

Yajun Wang<sup>†</sup>

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## Abstract

We study the matroid secretary problem when the valuation is defined by a submodular function. In this problem, the elements of the ground set of a matroid are arriving in random order. When one element arrives, we have to make an immediate and irrevocable decision regarding whether or not to accept it. Our objective is to select an independent set under the given matroid, so that the valuation of the set is maximized with a submodular function.

Our major result is a constant-competitive algorithm for the submodular matroid secretary problem under transversal matroids with monotonically increasing submodular valuation functions. In addressing it, we instead develop a constant competitive algorithm for a more general online bipartite graph matching problem. As a natural special case, we also consider the secretary problem under partition matroids and provide an extremely simple algorithm which can handle non-monotone submodular functions. This algorithm significantly improves the competitive ratio over the previous work.

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\*Shanghai Jiao Tong University. This work was done when the author was visiting Microsoft Research Asia.

<sup>†</sup>Microsoft Research Asia, [yajunw@microsoft.com](mailto:yajunw@microsoft.com)

# 1 Introduction

Secretary problems study the algorithms to accept elements with largest weights under certain constraints. The elements are arriving in random order and we have to make an immediate and irrevocable decision to whether take the current element as output or not. In the classical secretary problem [8, 11, 12], one interviewer is interviewing  $n$  candidates for a secretary position. The candidates are arriving in random order and the interviewer has to decide whether or not to hire the current candidate when she arrives. The goal is to hire the best secretary.

Notice that if the arriving order of the candidates is adversarially chosen, one cannot hope for an algorithm with reasonable chance to hire the best secretary. However, the random order assumption in the secretary problem setting is essential for practical solutions. In particular, for the classical secretary problem, we can observe first  $n/e$  candidates without hiring anyone. For subsequent candidates, we immediately hire the current candidate if she is better than anyone we have seen so far. It is not difficult to show that this mechanism will successfully hire the best secretary with probability  $1/e$ , if the candidates are arriving in random order.

Recently, Babaioff et al. [3] formulated the matroid secretary problem. Instead of hiring one candidate (element), the matroid secretary problem seeks to select a set of elements which form an independent set in a matroid. Again, the elements are arriving in random order and the weights of the elements are revealed when they arrive. Our objective is to maximize the total weights of the selected elements. They gave an  $O(\log r)$ -competitive algorithm for a general matroid, i.e. the expected total weights of the elements selected by the algorithm is  $\Omega(1/\log r)$  of the optimal solution. They conjectured that any matroid secretary problem allows a constant competitive algorithm. This conjecture is still widely open, while constant competitive algorithms have been found for various matroids: uniform/partition matroids[2, 16], truncated partition matroids[3], graphical matroids[1, 17], transversal matroids[7, 17] and laminar matroids[15].

In the matroid secretary problem, the weights are associated to elements and we aim to pick elements with large weights. In some scenarios, it is more natural to measure the quality of a set by a valuation function, which is not necessarily linearly additive. One set of functions widely used in the optimization community is the *submodular* functions. Such functions are characterized as functions with diminishing returns. Let  $[n] = \{1, 2, \dots, n\}$  be the set of total elements. A function  $f : 2^{[n]} \Rightarrow R$  is submodular if  $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$  for  $S, T \subseteq [n]$ . For an alternative definition, a function  $f$  is submodular iff for any  $S \subseteq T \subseteq [n]$  and element  $x \in [n]$ ,  $f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T)$ .

It is natural to extend the matroid secretary problem with submodular functions. In other words, the weights are not directly associated with elements. Instead, there exists an oracle to query the function value for any subset of the elements we have seen. Gupta et al. [14] studied the *non-monotone* submodular matroid maximization problem for both offline and online (secretary) versions, i.e. to find an independent set  $S$  such that  $f(S)$  is large where  $f(\cdot)$  is a given (possibly non-monotone) submodular function. For the online (secretary) version, they provided a  $O(\log r)$ -competitive algorithm for general matroids and a constant competitive algorithm for uniform matroids (algorithms achieving constant competitive ratios are obtained independently by Mohammond[4] et al.) and partition matroids where only one element can be selected from each group. The submodular function poses serious challenges in developing constant competitive algorithms. In particular, in most cases, we are working with marginal valuation functions, which depend on the set of elements has been selected. Such dependencies greatly complicate the analysis, and have to be addressed very carefully.

**Our results.** We provide constant competitive algorithms for the submodular matroid secretary problem for transversal matroids and partition matroids.

**Theorem 1.1** *There is a 48-competitive algorithm for the matroid secretary problem under a transversal matroid with a monotonically increasing submodular valuation function.*

This algorithm is developed to solve a more general bipartite vertex-at-a-time matching problem with a submodular valuation function on the set of edges. Notice that partition matroids are special cases of transversal matroids. So our result for the transversal matroids already provides an improvement comparing with the algorithm in [14]. However, this algorithm cannot handle non-monotone submodular functions.

We then develop an algorithm for the non-monotone submodular secretary problem under partition matroids, which achieves a significantly better competitive ratio.

**Theorem 1.2** *There is an 18-competitive algorithm for the submodular matroid secretary problem under a partition matroid. When the valuation function is monotonically increasing, the competitive ratio can be improved to 5.*

Our result for the transversal matroid case can be extended to the online matching problem in hypergraphs as in [17]. The algorithm for the partition matroid can be extended to the case that one can accept a fixed number of elements from each group, using techniques in [4]. Also, it is straightforward to solve the case for graphical matroids.

**Some techniques.** With a submodular valuation function  $f(\cdot)$ , the previously accepted elements will affect the valuation of the arriving ones. In this paper, however, we show that one can treat the submodular function “obliviously” in the cases of the transversal and partition matroids. In fact, our algorithms are simple adaptations from the corresponding algorithms with linearly additive valuation functions. On the other hand, the analysis in the submodular case requires new techniques.

For the transversal matroids, we simulate an online algorithm with a randomized greedy algorithm in Algorithm 3. The randomized greedy algorithm greedily selects elements with highest marginal value with respect to  $M$ , which is empty at first. However, with half probability, the element may either go to  $M$  or  $N$ , where  $N$  is the set of candidates our algorithm would accept. It is natural to expect that  $f(M)$  is large. Our result implies, on the other hand,  $f(N) = \Omega(f(M))$ . This is rather surprising, since the greedy algorithm is always choosing elements with good marginal value respect to  $M$ ! To achieve this, we examine the function  $f(N) + (f(N) + f(M) - f(M \cup N))$ .

<sup>1</sup> The second term can be viewed as the intersection of  $M$  and  $N$ . In particular, we show that in each step of the randomized greedy algorithm, either  $f(N)$  grows or  $f(N) + f(M) - f(M \cup N)$  grows in expectation. This observation although simple, is critical in our analysis and we believe it might be useful to further understand the submodular secretary problem under other matroid constraints.

For the partition matroids, our algorithm is running  $r$  parallel *classical secretary* algorithms on each group of elements. The valuation function is the marginal valuation function with respect to the current set of accepted elements. This is counter-intuitive since the decision for the elements in one group depends critically on the decisions on other groups. Nevertheless, our analysis shows a constant competitive ratio for this simple algorithm.

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<sup>1</sup>In the analysis, we inspect the set  $S \subset N$  instead.

In the analysis, we define the partial permutation set  $\mathcal{X}_{-i}$  to group permutations that have the same positions for elements in all groups other than group  $i$ . This grouping is extremely useful to decouple the dependencies between the permutation and the marginal valuation function used in accepting one element from group  $i$ . We show the element in group  $i$  that our algorithm accepts is expected (over a particular permutation group in  $\mathcal{X}_{-i}$ ) to be “good” comparing with the element of the optimal solution in group  $i$  with respect to the previously accepted set of elements, which itself is a random variable. Therefore, overall, the set of elements accepted by our algorithm is good.

**Related work.** The secretary problem has been studied decades ago, which is first published in [12] and has been folklore even earlier [9]. Motivated by the simple optimal solution for the classical secretary problem, several results have appeared for solving more complicated problems in this random permutation model. Usually, the goal is to find an independent set which maximizes the total weight. For example, Kleinberg [16] gave a  $1 + O(1/\sqrt{k})$ -competitive algorithm for selecting at most  $k$  elements to maximize the sum of the weights. Babaioff et al. [2] considered the Knapsack secretary problem, in which each element has weight and size, and the objective is to find a set of elements whose total size is at most a given integer such that the total weights are maximized. They gave a constant competitive algorithms. Random permutation model is also considered in other problems to improve the solution feasibility, e.g. online ads problem [6, 13], robust stream model [5].

Babaioff et al. [3] systematically introduced the *matroid secretary problem*, in which the solution has to be an independent set of a matroid and the goal is to find the independent set with maximum total weights. As we mentioned earlier, various special cases admit constant competitive ratios. The major open problem is to find a constant competitive algorithm for general matroids, if there exists.

The *submodular matroid secretary problem* [4, 14] introduces another freedom in the matroid secretary problem by allowing the evaluation function to be submodular instead of simply linearly additive. Constant competitive algorithms have been developed for the uniform matroids and the partition matroids.

## 2 Preliminary

In the matroid secretary problem, we assume the elements are arriving in random order. Our algorithms have to decide whether to take the current element or not when it arrives. The decision cannot be revoked and the set of accepted elements has to form an independent set in the given matroid.

### 2.1 Submodular functions

In this paper, we assume the quality of the solution is measured by a submodular function. Notice that throughout this paper, we only work with *non-negative* functions. For transversal matroids, we assume the submodular function is monotonically increasing. Our algorithm for partition matroids works with non-monotone submodular functions as well.

**Definition 2.1** Let  $U$  be the ground set. Let  $f(\cdot) : 2^U \rightarrow \mathbb{R}$  be a function mapping any subset of  $U$  to a real number.  $f(\cdot)$  is a submodular function if:

$$\forall S, T \subseteq U, f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

For simplicity, for any set  $S \subseteq U$ , we define  $f_S(\cdot)$  as follows. For any  $T \subseteq U$ ,  $f_S(T) = f(S \cup T) - f(S)$ . It is not difficult to see that  $f_S(\cdot)$  is submodular if  $f(\cdot)$  is submodular. For simplicity, when  $T = \{t\}$  is a singleton, we also write  $f(t) = f(\{t\})$ .

## 2.2 Matroids

In the matroid secretary problem, the set of accepted elements must form an independent set defined by a given matroid.

**Definition 2.2 (Matroids)** Let  $U \neq \emptyset$  be the ground set and  $\mathcal{I}$  be a set of subsets of  $U$ . The system  $\mathcal{M} = (U, \mathcal{I})$  is a matroid with independent sets  $\mathcal{I}$  if:

1. If  $A \subseteq B \subseteq U$  and  $B \in \mathcal{I}$ , then  $A \in \mathcal{I}$ .
2. If  $A, B \in \mathcal{I}$  and  $|A| < |B|$ , there exists an element  $x \in B \setminus A$  such that  $A \cup \{x\} \in \mathcal{I}$ .

In this paper, we work with the following two matroids.

**Definition 2.3 (Transversal matroids)** Let  $G = (L, R, E)$  be an undirected bipartite graph with left nodes in  $L$ , right nodes in  $R$  and edges in  $E$ . In the transversal matroid defined by  $G$ , the ground set is  $L$  and a set of left nodes  $S \subseteq L$  is independent if there exists a matching in  $G$  such that the set of left nodes in the matching is  $S$ .

**Definition 2.4 (Partition matroids)** Let  $U = U_1 \cup U_2 \cup \dots \cup U_r$  be the ground set with disjoint subsets  $U_i$  and  $n_i = |U_i|$ . In the partition matroid defined by  $\mathcal{M} = (U, \mathcal{I})$ ,  $S \subseteq U$  is independent if  $\forall i \in [r]$ ,  $|S \cap U_i| \leq 1$ .

## 2.3 Submodular Bipartite Vertex-a-time Matching Problem

The transversal matroid is defined on matchings of a bipartite graph. Korula and Pál [17] generalized the transversal matroid secretary problem to an online bipartite graph matching problem, motivated by [7]. We further generalize to submodular valuation functions. In particular, we introduce the *Submodular Bipartite Vertex-at-a-time Matching* (SBVM) problem.

In the SBVM problem, there is an underlying bipartite graph  $G(L \cup R, E)$ . We are given the set of right nodes  $R$ . The nodes in  $L$ , however, are arriving sequentially in *random order*. When a vertex  $\ell \in L$  arrives, all edges incident to  $\ell$  are revealed. We assume the availability of an oracle for the submodular valuation function, which we can query the function value for any subsets of the edges we have seen. We must immediately decide to accept an edge to match  $\ell$  with a vertex of  $R$  or drop all edges incident to  $\ell$ .

We claim that the matroid secretary problem under a transversal matroid is a special case of the SBVM problem, when the valuation function is submodular. In particular, the valuation on  $L$  in the transversal matroid can be extended to the valuation on the edges  $E$ . Let  $f(\cdot)$  be a function defined on the subsets of  $L$ . We define a function  $g(\cdot)$  on the subsets of  $E$  as follows: for  $E' \subseteq E$ ,  $g(E') = f(L \cap E')$ , where  $L \cap E'$  is the set of left nodes incident to  $E'$ .<sup>2</sup>

**Lemma 2.5** *If  $f(\cdot)$  is a monotonically increasing submodular function,  $g(\cdot)$  is monotonically increasing submodular.*

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<sup>2</sup>The ties in the valuation function have to be broken in a consistent way.

Hence, for the submodular matroid secretary problem under a transversal matroid, we can extend the valuation function on  $L$  to the set of edges in the underlying bipartite graph. The optimum solutions for both problems are the same. In fact, if we find a matching, which is a good approximation of the SBVM problem, the left nodes of the matching is a good approximation of the matroid secretary problem with the same approximation ratio.

### 3 Algorithm for the SBVM Problem

Notice that all submodular functions used in this section are monotonically increasing. Recall from Section 2 that the submodular matroid secretary problem under a transversal matroid is a special case of the submodular bipartite vertex-at-a-time (SBVM) problem. In this section, we show that an adapted algorithm from that of [7, 17] gives a competitive ratio of 48 for the SBVM problem. Our adaptation is based on the following algorithm GREEDY.

**Input:**  $G = (L, R, E)$  and function  $f(\cdot)$   
**Output:** Matching  $S$   
 $S \leftarrow \emptyset$ ;  
**while**  $\{e \mid S \cup \{e\} \text{ is a matching}\} \neq \emptyset$  **do**  
     $e^* = \arg \max_e \{f_S(e) \mid S \cup \{e\} \text{ is a matching}\}$ ;  
     $S \leftarrow S \cup e^*$ ;  
**end**  
return  $S$ ;

**Algorithm 1:** GREEDY

**Lemma 3.1** *For a bipartite graph  $G(L, R, E)$  and a monotonically increasing submodular function  $f(\cdot) \geq 0$  defined on all subsets of  $E$ , GREEDY is a 3-approximate algorithm.*

We now present our algorithm ONLINE for the SBVM problem. When  $f(\cdot)$  is linearly additive, this algorithm is identical to the algorithms in [7, 17]. But in general submodular case, our algorithm based on GREEDY is simpler and clearer .

In the algorithm, we first observe  $k = \text{Binom}(|L|, 1/2)$  number of left nodes  $H$  while rejecting all edges incident to them, where  $k$  is the binomial random variable. We build the matching  $M$  by running GREEDY on the edges  $E \cap (H \times R)$ . For any subsequent left node  $\ell$ , again we run GREEDY on the edges  $E \cap (H \cup \{\ell\} \times R)$  to build a new matching  $M_\ell$ . If there exists an edge  $e_\ell$  incident to  $\ell$  in  $M_\ell$ , we add  $e_\ell$  into  $N$ , which is a set maintained for later analysis. The edge  $e_\ell$  is accepted by our algorithm if it forms a matching with the set of accepted edges so far.

Korula and Pál [17] observed that there is a randomized offline algorithm which can simulate their online algorithm. In our case, we also describe a closely related algorithm SIMULATE as Algorithm 3. SIMULATE is easier to analyze, since it simulates the randomness of the input in ONLINE by its internal random coins. In SIMULATE, we have a reference matching  $M$ , a working set  $N$  and a result set  $S \subseteq N$ .

We start with two empty sets  $M$  and  $N$ . At each time, we pick an edge  $e = (\ell, r)$  greedily with respect to function  $f_M(\cdot)$  such that  $M \cup \{e\}$  is a matching. We then flip a coin. If the coin is head,  $e$  is added to  $M$ . Otherwise, it is added to  $N$ . In both cases, we remove all edges incident to  $\ell$ . The algorithm stops when no edge can be picked.

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Input:  $G = (L, R, E)$  and function  $f(\cdot)$  on  $E$ .
Output: Matching ALG.
 $k \leftarrow \text{Binom}(|L|, \frac{1}{2})$ ;
Observe  $H \leftarrow$  the first  $k$  vertices of  $L$  and reject all edges incident to  $H$ ;
 $E' \leftarrow E \cap (H \times R)$ ;
 $M \leftarrow \text{GREEDY}(G(H, R, E'))$ ;
 $\text{ALG}, N \leftarrow \emptyset$ ;
for each subsequent  $\ell \in L \setminus H$  do
     $E_\ell \leftarrow E \cap (H \cup \{\ell\} \times R)$ ;
     $M_\ell \leftarrow \text{GREEDY}(G(H \cup \{\ell\}, R, E_\ell))$ ;
    if  $M_\ell \neq M$  then
        Let  $e_\ell$  be the edge in  $M_\ell$  incident to  $\ell$ ;
         $N = N \cup \{e_\ell\}$ ;
        if  $\text{ALG} \cup \{e_\ell\}$  is a matching then
             $\text{ALG} \leftarrow \text{ALG} \cup \{e_\ell\}$ ;
            Accept  $e_\ell$ ;
            Continue;
        end
        Reject all edges incident to  $\ell$ ;
    end
end

```

**Algorithm 2:** ONLINE for the SBVM Problem

We introduce the two observations about SIMULATE from [17]: Once any edge incident to a vertex  $\ell \in L$  has been picked, no other edge incident to  $\ell$  will be picked later since we remove all of them. Second, multiple edges incident to  $r \in R$  might be picked until one of these edges is added to  $M$ . As a consequence, at the end of SIMULATE,  $M$  is a matching, but  $N$  might not be.

Our final pruning step takes care the case that  $N$  is not a matching, i.e., there are multiple edges incident to the same vertex of  $R$  in  $N$ . The final output is a matching  $S$ . We will prove that  $S$  is a constant approximation. Before that, we justify the usefulness of SIMULATE by the following lemma, which is implicitly assumed in [17]. We give a proof in the appendix for completeness.

**Lemma 3.2** *The sets of edges of  $M$  and  $N$  by SIMULATE has the same joint distribution as the  $M$  and  $N$  generated by ONLINE with a random permutation of the left nodes  $L$ .*

### 3.1 Analysis of ONLINE and SIMULATE

Let OPT be the optimal matching in  $G$ . We first study the performance of  $M$ . The following lemma is very intuitive.

**Lemma 3.3**  $\mathbb{E}[f(M)] \geq \frac{1}{6}f(\text{OPT})$  .

Let  $E_s$  be the set of edges picked in SIMULATE. Define  $M_e$  be the set of edges in  $M$  when  $e$  is picked. For each node  $r \in R$ , define  $e_r \in E_s$  as the first edge incident to  $r$  picked in SIMULATE; define  $m_r \in M$  as the edge incident to  $r$ . By the submodularity of  $f(\cdot)$  and the greedy algorithm,



```

Input:  $G = (L, R, E)$  and function  $f(\cdot)$ 
Output: The set of selected edges  $S$ 
 $M, N, S \leftarrow \emptyset$ ;
while  $\exists e^* = (\ell^*, r^*) = \arg \max_{e \in E} \{f_M(e) \mid M \cup \{e\} \text{ is a matching}\}$  do
    | Flip a coin with probability  $\frac{1}{2}$  of head;
    | if head then  $M \leftarrow M \cup \{e\}$ ;
    | else  $N \leftarrow N \cup \{e\}$ ;
    | Remove all edges incident to  $\ell^*$  from  $E$ ;
end
foreach edge  $e = (\ell, r) \in N$  do
    | Add  $e$  to  $S$  if  $e$  is the only edge incident to  $r$  in  $N$ ;
end
return  $S$ ;

```

**Algorithm 3:** SIMULATE

we have  $f_{M_{e_r}}(e_r) \geq f_{M_{m_r}}(m_r)$ . (Both  $e_r$  and  $m_r$  are random edges, which may not exist. We define the  $f_T(e) = 0$  when  $e$  does not exist.) Since  $f(M) = \sum_{r \in R} f_{M_{m_r}}(m_r)$ , the following proposition can be directly implied.

**Proposition 3.4**  $\sum_{r \in R} f_{M_{e_r}}(e_r) \geq f(M)$ .

In order to show the performance of  $f(S)$ , it is sufficient to compare with  $\sum_{r \in R} f_{M_{e_r}}(e_r)$ . However, it is not even intuitively clear why there exists a relationship between these two quantities. In particular, recall that in SIMULATE, we always pick edges greedily *respect to the current  $M$* , regardless of  $N$  and  $S$ .

Instead, we inspect the function  $F(M, S) = f(S) + (f(M) + f(S) - f(M \cup S))$  during the execution of SIMULATE. The second term  $f(M) + f(S) - f(M \cup S)$  can be viewed as the intersection of  $M$  and  $S$ . Intuitively, when we pick an edge  $e$  in SIMULATE, if  $f_M(e)$  and  $f_S(e)$  are comparable to each other, the growth of  $f(S)$  is good. On the other hand, in case that  $f_S(e) \ll f_M(e)$ , with probability  $1/2$ ,  $e$  is added into  $M$ , in which case the “intersection” between  $M$  and  $S$  grows. The proof of the following lemma concretely implements this idea.

**Lemma 3.5**  $\mathbb{E}[f(S)] \geq \frac{1}{8} \mathbb{E} [\sum_{r \in R} f_{M_{e_r}}(e_r)]$

**Proof:** Consider the function  $F(M, S)$ . Let  $M_r$  and  $S_r$  be the set of edges in  $M$  and  $S$  respectively when the edge  $e_r$  is picked in SIMULATE. Denote  $M'_r$  and  $S'_r$  as the set of edges in  $M$  and  $S$  after  $e_r$  is processed.

Define  $\Delta_r = F(M'_r, S'_r) - F(M_r, S_r)$ .  $F(M, S)$  is monotonically increasing when edges are processed in SIMULATE. Therefore,  $F(M, S) \geq \sum_{r \in R} \Delta_r$ . Let  $\mathcal{F}_r$  be the sub- $\sigma$ -algebra encoding all randomness up to the time  $e_r$  is picked in SIMULATE. Notice that  $M_r$ ,  $S_r$  and  $e_r$  are  $\mathcal{F}_r$  measurable.



$$\begin{aligned}
\mathbb{E}[\Delta_r \mid \mathcal{F}_r] &= \Pr[e_r \in S \mid \mathcal{F}_r](2f_{S_r}(e_r) - f_{M_r \cup S_r}(e_r)) + \Pr[e_r \in M \mid \mathcal{F}_r](f_{M_r}(e_r) - f_{M_r \cup S_r}(e_r)) \\
&\geq \frac{1}{4}(2f_{S_r}(e_r) - f_{M_r \cup S_r}(e_r)) + \frac{1}{2}(f_{M_r}(e_r) - f_{M_r \cup S_r}(e_r)) \\
&\geq \frac{1}{2}f_{M_r}(e_r) - \frac{1}{4}f_{M_r \cup S_r}(e_r) \\
&\geq \frac{1}{4}f_{M_r}(e_r).
\end{aligned}$$

The last two inequalities come from the submodularity of  $f(\cdot)$ . Therefore,

$$2\mathbb{E}[f(S)] \geq \mathbb{E}[F(M, S)] = \sum_{r \in R} \mathbb{E}[\Delta_r] = \sum_{r \in R} \mathbb{E}_{\mathcal{F}_r}[\mathbb{E}[\Delta_r \mid \mathcal{F}_r]] \geq \frac{1}{4} \sum_{r \in R} \mathbb{E}[f_{M_r}(e_r)]$$

□

Combing the analysis above, we have the following theorem.

**Theorem 3.6** *There are 48-competitive algorithms for the SBVM problem and the submodular matroid secretary problem under a transversal matroid with monotonically increasing submodular valuation functions.*

## 4 Algorithms for the partition matroids

In this section, we develop constant competitive algorithms for the submodular matroid secretary problem under partition matroids. Recall that an independent set in a partition matroid contains at most one element from each group.

### 4.1 Monotone submodular functions

We first discuss the case when  $f(\cdot)$  is a monotonically increasing submodular function. Our algorithm is extremely simple, though the analysis for its performance is subtle to obtain. In fact, we parallel run  $r$  classical secretary algorithms on each group of elements in PARTITION.

Let  $\text{Id}(\cdot)$  be the function which returns the id of the group for a particular element. For group  $i$ , we observe the first  $n_i/2$  elements and reject all of them. For each sub-sequential element in  $U_i$ , we accept it if it is better than all elements we have seen in  $U_i$ , with respect to  $f_S(\cdot)$  where  $S$  is the current set of accepted elements. (If the marginal value is negative, we do not accept the current element and any element from this group.)

Notice that  $S$  is changing in our algorithm. Consider a particular group  $i$  and current  $S$ . It could be the case that we have missed the “best element” in group  $i$  respect to  $f_S(\cdot)$  simply because  $S$  was different when it arrived. So it is rather counter-intuitive that this algorithm is a constant competitive algorithm.

We require more notations. Let  $\text{OPT} = \arg \max\{f(T) \mid T \in \mathcal{I}\}$  be the optimal solution and  $S \in \mathcal{I}$  be the output of our algorithm. For a given permutation  $p$  of elements in  $U$ , let  $S^p$  be the output of our algorithm. Denote  $\text{OPT}_i$  (resp.  $e_i, e_i^p$ ) for  $i \in [r]$  as the element of  $\text{OPT}$  (resp.  $S, S^p$ ) in group  $i$ . Let  $S_i$  (resp.  $S_i^p$ ) be the set of elements accepted by our algorithm when one of the  $b_i^j$ s is set to true in PARTITION. If all  $b_i^j$  are false after the algorithm terminates, we set  $S_i = S$ .

```

Input: Partition matroid  $\mathcal{M} = (U, \mathcal{I})$ , function  $f(\cdot)$ 
Output: The set of selected elements  $S \in \mathcal{I}$ 
 $\forall i \in [r], T_i \leftarrow \emptyset, \forall j \in [n_i], b_i^j \leftarrow \text{false}; S \leftarrow \emptyset;$ 
for each element  $e \in U$  do
     $T_{\text{Id}(e)} \leftarrow T_{\text{Id}(e)} \cup \{e\};$ 
    if  $(\forall j \in [n_{\text{Id}(e)}], b_{\text{Id}(e)}^j \text{ is false}) \wedge (|T_{\text{Id}(e)}| > \frac{n_{\text{Id}(e)}}{2}) \wedge (e = \arg \max_{e' \in T_{\text{Id}(e)}} \{f_S(e')\})$  then
         $b_{\text{Id}(e)}^{|T_{\text{Id}(e)}|} \leftarrow \text{true};$ 
        if  $f_S(e) > 0$  then
             $S \leftarrow S \cup \{e\}; \text{Accept } e;$ 
        end
    end
end
return  $S;$ 

```

**Algorithm 4:** PARTITION

**Remark on Non-existence.** In general,  $e_i$  and  $e_i^p$  may not exist. We evaluate  $f_{S_i}(e_i)$  and  $f_{S_i^p}(e_i^p)$  to 0 in this case. When  $f(\cdot)$  is not monotone,  $\text{OPT}_i$  may not exist as well, in which case  $f_T(\text{OPT}_i) = 0$  for any  $T \subseteq U$ .

Our analysis crucially depend on the following definition to group the permutations of the arriving elements.

**Definition 4.1 (Permutation group)** For a permutation  $p$  of elements  $U$ , define  $P_i(p)$  be the positions of  $U_i$  in  $p$ . For each  $i \in [r]$ , define the following equivalence relationship on permutations. For two permutations  $\mathbf{x} = \{x_j\}, \mathbf{y} = \{y_j\} \in \mathcal{P}$ , they are equivalent if and only if

$$P_i(\mathbf{x}) = P_i(\mathbf{y}) \text{ and } \forall j \notin P_i(\mathbf{x}), x_j = y_j.$$

Define  $\mathcal{X}_{-i}$  as the set of equivalence groups respect to  $i$ .

For permutations in the same group in  $\mathcal{X}_{-i}$ , they share the same positions for all elements except  $U_i$ . Define  $[p]_i \in \mathcal{X}_{-i}$  as the permutation group respect to  $i$  containing the permutation  $p$ . When the context is clear, we drop the subscript in  $[p]_i$  and write  $[p]$  as we always consider the group  $i$ .

For a permutation  $p$ , define the positions of  $U_i$  in  $p$  as  $\{\ell_1, \dots, \ell_{n_i}\}$  in increasing order. Now define  $T_j^p$  be the set of elements accepted by our algorithm before  $\ell_j$ th element arrives in  $p$ , if we force our algorithm to reject all elements in  $U_i$ . By definition, for any permutation  $q \in [p]$ ,  $T_j^q$  is identical to each other. Therefore, we write  $T_j^{[p]} = T_j^q$  for  $q \in [p]$ . We also define  $T^{[p]}$  be the elements our algorithm accepts with a permutation in  $[p]$ , if we force our algorithm to reject all elements in  $U_i$ .

We are interested in the probability that our algorithm accepts an element in group  $i$  as well as the quality of the element when it is accepted. The following lemma is critical in this section, which captures the interesting cases.

**Lemma 4.2** *Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group, where  $x$  is a permutation. Let  $P_i(x) = \{\ell_1, \ell_2, \dots, \ell_{n_i}\}$  be the set of positions for elements in  $U_i$  in increasing order. Let  $e_i^j$  be the element accepted by our algorithm at position  $\ell_j$ . We have*

1.  $\forall j \in [n_i], \Pr[f(e_i^j) \geq \max_{e \in U_i} \{f_{T_j^{[x]}}(e)\} \mid p \in [x]] = \frac{1}{2^{j-1}}$
2.  $\forall j \in [n_i], \Pr[b_i^j \text{ is true} \mid p \in [x]] = \frac{n_i}{2^{j(j-1)}}$
3.  $\Pr[\forall j \in [n_i], b_i^j \text{ is false} \mid p \in [x]] = \frac{1}{2}$

In the first and the second cases above, we mark  $b_i^j$  as true which implies  $S_i^p = T_j^{[x]}$  by definition. In the third case,  $S_i^p = T^{[x]}$ . The following lemma compares the performance of  $\text{OPT}_i$  with  $e_i$  respect to  $S_i^p$ . Notice that both  $S_i^p$  and  $e_i$  are random variables.

**Lemma 4.3** *Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group. Assume the elements are arriving in a random permutation  $p$ . We have*

$$\mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x]] \geq \frac{1}{4} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x]].$$

It is then straightforward to bound the performance of  $f(S)$ .

**Lemma 4.4**  $\mathbb{E}[f(S)] \geq \frac{1}{5} \mathbb{E}[f(S \cup \text{OPT})]$

When  $f(\cdot)$  is monotonically increasing, we already have a constant competitive algorithm.

**Theorem 4.5** *There is a 5-competitive algorithm for the matroid secretary problem under a partition matroid when the valuation function is a monotonically increasing submodular function.*

## 4.2 Non-monotone submodular functions

We have shown that PARTITION is 5-competitive when  $f(\cdot)$  is a monotonically increasing submodular function. Notice that, all the results in the previous section can be carried over to the case with non-monotone submodular functions. In order to achieve constant competitive ratio in the non-monotone case, we need the following simple result. Notice that we always assume  $f(\cdot) \geq 0$ .

**Proposition 4.6** *Let  $f(\cdot)$  be a non-negative submodular function. For any sets  $A, B, C \subseteq U$  such that  $A \cap B = \emptyset$ , we have:*

$$f(A \cup C) + f(B \cup C) \geq f(C).$$

The proof is very simple. Notice that  $(A \cup C) \cap (B \cup C) = C$  as  $A \cap B = \emptyset$ . The inequality comes from the submodularity and non-negativeness of  $f(\cdot)$ .

**Algorithm.** We first decide to run our algorithm PARTITION in mode  $A$  or  $B$  by a random choice. If we are in mode  $A$ , when we want to accept an element  $e$  in group  $i$ , we flip a coin. If the coin is head, we accept the element. Otherwise, we reject this element and never accept any element in group  $i$  in the future. Symmetrically, we do the opposite in mode  $B$ , i.e., only accept the element if the coin is tail.

Our modification simplifies the process in [14]. Denote  $\text{ALG}_A$  (resp.  $\text{ALG}_B$ ) as the output of our algorithm in mode  $A$  (resp.  $B$ ). Notice that  $\text{ALG}_A$  and  $\text{ALG}_B$  have the same distribution and  $\text{ALG}_A \cap \text{ALG}_B = \emptyset$ . So it is sufficient to show

$$\mathbb{E}[\text{ALG}_A] \geq \frac{1}{9}\mathbb{E}[\text{ALG}_A \cup \text{OPT}]. \quad (1)$$

Detailed proofs are left in the appendix. Notice that our algorithm will randomly choose  $\text{ALG}_A$  or  $\text{ALG}_B$  based on the random choice before any element arrives. Clearly, this is an 18-competitive algorithm.

**Theorem 4.7** *There is an 18-competitive algorithm for the matroid secretary problem under a partition matroid when the valuation function is a submodular function.*

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## A Introduction

Secretary problems study the algorithms to accept elements with largest weights under certain constraints. The elements are arriving in random order and we have to make an immediate and irrevocable decision to whether take the current element as output or not. In the classical secretary problem [8, 11, 12], one interviewer is interviewing  $n$  candidates for a secretary position. The candidates are arriving in random order and the interviewer has to decide whether or not to hire the current candidate when she arrives. The goal is to hire the best secretary.

Notice that if the arriving order of the candidates is adversarially chosen, one cannot hope for an algorithm with reasonable chance to hire the best secretary. However, the random order assumption in the secretary problem setting is essential for practical solutions. In particular, for the classical secretary problem, we can observe first  $n/e$  candidates without hiring anyone. For subsequent candidates, we immediately hire the current candidate if she is better than anyone we have seen so far. It is not difficult to show that this mechanism will successfully hire the best secretary with probability  $1/e$ , if the candidates are arriving in random order.

Recently, Babaioff et al. [3] formulated the matroid secretary problem. Instead of hiring one candidate (element), the matroid secretary problem seeks to select a set of elements which form an independent set in a matroid. Again, the elements are arriving in random order and the weights of the elements are revealed when they arrive. Our objective is to maximize the total weights of the selected elements. They gave an  $O(\log r)$ -competitive algorithm for a general matroid, i.e. the expected total weights of the elements selected by the algorithm is  $\Omega(1/\log r)$  of the optimal solution. They conjectured that any matroid secretary problem allows a constant competitive algorithm. This conjecture is still widely open, while constant competitive algorithms have been found for various matroids: uniform/partition matroids[2, 16], truncated partition matroids[3], graphical matroids[1, 17], transversal matroids[7, 17] and laminar matroids[15].

In the matroid secretary problem, the weights are associated to elements and we aim to pick elements with large weights. In some scenarios, it is more natural to measure the quality of a set by a valuation function, which is not necessarily linearly additive. One set of functions widely used in the optimization community is the *submodular* functions. Such functions are characterized as functions with diminishing returns. Let  $[n] = \{1, 2, \dots, n\}$  be the set of total elements. A function  $f : 2^{[n]} \Rightarrow R$  is submodular if  $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$  for  $S, T \subseteq [n]$ . For an alternative definition, a function  $f$  is submodular iff for any  $S \subseteq T \subseteq [n]$  and element  $x \in [n]$ ,  $f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T)$ .

It is natural to extend the matroid secretary problem with submodular functions. In other words, the weights are not directly associated with elements. Instead, there exists an oracle to query the function value for any subset of the elements we have seen. Gupta et al. [14] studied the *non-monotone* submodular matroid maximization problem for both offline and online (secretary) versions, i.e. to find an independent set  $S$  such that  $f(S)$  is large where  $f(\cdot)$  is a given (possibly non-monotone) submodular function. For the online (secretary) version, they provided a  $O(\log r)$ -competitive algorithm for general matroids and a constant competitive algorithm for uniform matroids (algorithms achieving constant competitive ratios are obtained independently by Mohammond[4] et al.) and partition matroids where only one element can be selected from each group. The submodular function poses serious challenges in developing constant competitive algorithms. In particular, in most cases, we are working with marginal valuation functions, which depend on the set of elements has been selected. Such dependencies greatly complicate the analysis, and have to be addressed very carefully.



**Our results.** We provide constant competitive algorithms for the submodular matroid secretary problem for transversal matroids and partition matroids.

**Theorem A.1** *There is a 48-competitive algorithm for the matroid secretary problem under a transversal matroid with a monotonically increasing submodular valuation function.*

This algorithm is developed to solve a more general bipartite vertex-at-a-time matching problem with a submodular valuation function on the set of edges. Notice that partition matroids are special cases of transversal matroids. So our result for the transversal matroids already provides an improvement comparing with the algorithm in [14]. However, this algorithm cannot handle non-monotone submodular functions.

We then develop an algorithm for the non-monotone submodular secretary problem under partition matroids, which achieves a significantly better competitive ratio.

**Theorem A.2** *There is an 18-competitive algorithm for the submodular matroid secretary problem under a partition matroid. When the valuation function is monotonically increasing, the competitive ratio can be improved to 5.*

Our result for the transversal matroid case can be extended to the online matching problem in hypergraphs as in [17]. The algorithm for the partition matroid can be extended to the case that one can accept a fixed number of elements from each group, using techniques in [4]. Also, it is straightforward to solve the case for graphical matroids.

**Some techniques.** With a submodular valuation function  $f(\cdot)$ , the previously accepted elements will affect the valuation of the arriving ones. In this paper, however, we show that one can treat the submodular function “obliviously” in the cases of the transversal and partition matroids. In fact, our algorithms are simple adaptations from the corresponding algorithms with linearly additive valuation functions. On the other hand, the analysis in the submodular case requires new techniques.

For the transversal matroids, we simulate an online algorithm with a randomized greedy algorithm in Algorithm 3. The randomized greedy algorithm greedily selects elements with highest marginal value with respect to  $M$ , which is empty at first. However, with half probability, the element may either go to  $M$  or  $N$ , where  $N$  is the set of candidates our algorithm would accept. It is natural to expect that  $f(M)$  is large. Our result implies, on the other hand,  $f(N) = \Omega(f(M))$ . This is rather surprising, since the greedy algorithm is always choosing elements with good marginal value respect to  $M$ ! To achieve this, we examine the function  $f(N) + (f(N) + f(M) - f(M \cup N))$ .<sup>3</sup> The second term can be viewed as the intersection of  $M$  and  $N$ . In particular, we show that in each step of the randomized greedy algorithm, either  $f(N)$  grows or  $f(N) + f(M) - f(M \cup N)$  grows in expectation. This observation although simple, is critical in our analysis and we believe it might be useful to further understand the submodular secretary problem under other matroid constraints.

For the partition matroids, our algorithm is running  $r$  parallel *classical secretary* algorithms on each group of elements. The valuation function is the marginal valuation function with respect to the current set of accepted elements. This is counter-intuitive since the decision for the elements in one group depends critically on the decisions on other groups. Nevertheless, our analysis shows a constant competitive ratio for this simple algorithm.

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<sup>3</sup>In the analysis, we inspect the set  $S \subset N$  instead.

In the analysis, we define the partial permutation set  $\mathcal{X}_{-i}$  to group permutations that have the same positions for elements in all groups other than group  $i$ . This grouping is extremely useful to decouple the dependencies between the permutation and the marginal valuation function used in accepting one element from group  $i$ . We show the element in group  $i$  that our algorithm accepts is expected (over a particular permutation group in  $\mathcal{X}_{-i}$ ) to be “good” comparing with the element of the optimal solution in group  $i$  with respect to the previously accepted set of elements, which itself is a random variable. Therefore, overall, the set of elements accepted by our algorithm is good.

**Related work.** The secretary problem has been studied decades ago, which is first published in [12] and has been folklore even earlier [9]. Motivated by the simple optimal solution for the classical secretary problem, several results have appeared for solving more complicated problems in this random permutation model. Usually, the goal is to find an independent set which maximizes the total weight. For example, Kleinberg [16] gave a  $1 + O(1/\sqrt{k})$ -competitive algorithm for selecting at most  $k$  elements to maximize the sum of the weights. Babaioff et al. [2] considered the Knapsack secretary problem, in which each element has weight and size, and the objective is to find a set of elements whose total size is at most a given integer such that the total weights are maximized. They gave a constant competitive algorithms. Random permutation model is also considered in other problems to improve the solution feasibility, e.g. online ads problem [6, 13], robust stream model [5].

Babaioff et al. [3] systematically introduced the *matroid secretary problem*, in which the solution has to be an independent set of a matroid and the goal is to find the independent set with maximum total weights. As we mentioned earlier, various special cases admit constant competitive ratios. The major open problem is to find a constant competitive algorithm for general matroids, if there exists.

The *submodular matroid secretary problem* [4, 14] introduces another freedom in the matroid secretary problem by allowing the evaluation function to be submodular instead of simply linearly additive. Constant competitive algorithms have been developed for the uniform matroids and the partition matroids.

## B Preliminary

In the matroid secretary problem, we assume the elements are arriving in random order. Our algorithms have to decide whether to take the current element or not when it arrives. The decision cannot be revoked and the set of accepted elements has to form an independent set in the given matroid.

### B.1 Submodular functions

In this paper, we assume the quality of the solution is measured by a submodular function. Notice that throughout this paper, we only work with *non-negative* functions. For transversal matroids, we assume the submodular function is monotonically increasing. Our algorithm for partition matroids works with non-monotone submodular functions as well.

**Definition B.1** Let  $U$  be the ground set. Let  $f(\cdot) : 2^U \rightarrow \mathbb{R}$  be a function mapping any subset of  $U$  to a real number.  $f(\cdot)$  is a submodular function if:

$$\forall S, T \subseteq U, f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

For simplicity, for any set  $S \subseteq U$ , we define  $f_S(\cdot)$  as follows. For any  $T \subseteq U$ ,  $f_S(T) = f(S \cup T) - f(S)$ . It is not difficult to see that  $f_S(\cdot)$  is submodular if  $f(\cdot)$  is submodular. For simplicity, when  $T = \{t\}$  is a singleton, we also write  $f(t) = f(\{t\})$ .

## B.2 Matroids

In the matroid secretary problem, the set of accepted elements must form an independent set defined by a given matroid.

**Definition B.2 (Matroids)** Let  $U \neq \emptyset$  be the ground set and  $\mathcal{I}$  be a set of subsets of  $U$ . The system  $\mathcal{M} = (U, \mathcal{I})$  is a matroid with independent sets  $\mathcal{I}$  if:

1. If  $A \subseteq B \subseteq U$  and  $B \in \mathcal{I}$ , then  $A \in \mathcal{I}$ .
2. If  $A, B \in \mathcal{I}$  and  $|A| < |B|$ , there exists an element  $x \in B \setminus A$  such that  $A \cup \{x\} \in \mathcal{I}$ .

In this paper, we work with the following two matroids.

**Definition B.3 (Transversal matroids)** Let  $G = (L, R, E)$  be an undirected bipartite graph with left nodes in  $L$ , right nodes in  $R$  and edges in  $E$ . In the transversal matroid defined by  $G$ , the ground set is  $L$  and a set of left nodes  $S \subseteq L$  is independent if there exists a matching in  $G$  such that the set of left nodes in the matching is  $S$ .

**Definition B.4 (Partition matroids)** Let  $U = U_1 \cup U_2 \cup \dots \cup U_r$  be the ground set with disjoint subsets  $U_i$  and  $n_i = |U_i|$ . In the partition matroid defined by  $\mathcal{M} = (U, \mathcal{I})$ ,  $S \subseteq U$  is independent if  $\forall i \in [r]$ ,  $|S \cap U_i| \leq 1$ .

## B.3 Submodular Bipartite Vertex-a-time Matching Problem

The transversal matroid is defined on matchings of a bipartite graph. Korula and Pál [17] generalized the transversal matroid secretary problem to an online bipartite graph matching problem, motivated by [7]. We further generalize to submodular valuation functions. In particular, we introduce the *Submodular Bipartite Vertex-at-a-time Matching* (SBVM) problem.

In the SBVM problem, there is an underlying bipartite graph  $G(L \cup R, E)$ . We are given the set of right nodes  $R$ . The nodes in  $L$ , however, are arriving sequentially in *random order*. When a vertex  $\ell \in L$  arrives, all edges incident to  $\ell$  are revealed. We assume the availability of an oracle for the submodular valuation function, which we can query the function value for any subsets of the edges we have seen. We must immediately decide to accept an edge to match  $\ell$  with a vertex of  $R$  or drop all edges incident to  $\ell$ .

We claim that the matroid secretary problem under a transversal matroid is a special case of the SBVM problem, when the valuation function is submodular. In particular, the valuation on  $L$  in the transversal matroid can be extended to the valuation on the edges  $E$ . Let  $f(\cdot)$  be a function defined on the subsets of  $L$ . We define a function  $g(\cdot)$  on the subsets of  $E$  as follows: for  $E' \subseteq E$ ,  $g(E') = f(L \cap E')$ , where  $L \cap E'$  is the set of left nodes incident to  $E'$ .<sup>4</sup>

**Lemma B.5** *If  $f(\cdot)$  is a monotonically increasing submodular function,  $g(\cdot)$  is monotonically increasing submodular.*

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<sup>4</sup>The ties in the valuation function have to be broken in a consistent way.

**Proof:** Clearly, if  $f(\cdot)$  is monotonically increasing,  $g(\cdot)$  must be monotonically increasing. Let  $E'' \subseteq E' \subseteq E$ . We have  $E'' \cap L \subseteq E' \cap L$ . Therefore, for any edge  $e \in E$ , we want to show that

$$g(E'' \cup \{e\}) - g(E'') \geq g(E' \cap \{e\}) - g(E'). \quad (2)$$

If  $e$  is sharing the left node with  $E'$ , by monotonicity, the left term in Eq. 2 is non-negative while the right term is zero. So the statement is true. On the other hand, when  $e$  is not sharing the left node with  $E'$ ,  $e$  is not sharing the left node with  $E''$  either. Eq. 2 in this case comes directly from the submodularity of  $f(\cdot)$ .  $\square$

Hence, for the submodular matroid secretary problem under a transversal matroid, we can extend the valuation function on  $L$  to the set of edges in the underlying bipartite graph. The optimum solutions for both problems are the same. In fact, if we find a matching, which is a good approximation of the SBVM problem, the left nodes of the matching is a good approximation of the matroid secretary problem with the same approximation ratio.

## C Algorithm for the SBVM Problem

Notice that all submodular functions used in this section are monotonically increasing. Recall from Section B that the submodular matroid secretary problem under a transversal matroid is a special case of the submodular bipartite vertex-at-a-time (SBVM) problem. In this section, we show that an adapted algorithm from that of [7, 17] gives a competitive ratio of 48 for the SBVM problem. Our adaptation is based on the following algorithm GREEDY.

**Input:**  $G = (L, R, E)$  and function  $f(\cdot)$   
**Output:** Matching  $S$   
 $S \leftarrow \emptyset$ ;  
**while**  $\{e \mid S \cup \{e\} \text{ is a matching}\} \neq \emptyset$  **do**  
     $e^* = \arg \max_e \{f_S(e) \mid S \cup \{e\} \text{ is a matching}\}$ ;  
     $S \leftarrow S \cup e^*$ ;  
**end**  
return  $S$ ;

**Algorithm 5:** GREEDY

**Lemma C.1** *For a bipartite graph  $G(L, R, E)$  and a monotonically increasing submodular function  $f(\cdot) \geq 0$  defined on all subsets of  $E$ , GREEDY is a 3-approximate algorithm.*

**Proof:** First we show that all matchings of  $G = (L, R, E)$  can be represented by independent sets, which are the intersection of two partition matroids. Both ground sets of these two partition matroids are  $E$ . In the matroid  $M_1(E, \mathcal{I}_1)$  (resp.  $M_2(E, \mathcal{I}_2)$ ), a set of edges is independent if no two edges in it share a left (resp. right) node. It is easy to see that the set of matchings in  $G$  is exactly  $\mathcal{I}_1 \cap \mathcal{I}_2$ .

The Theorem 2.1 in [10] shows that the greedy algorithm is a  $p + 1$  approximation for the submodular function maximization problem under the intersection of  $p$  matroids. Therefore, our algorithm is a 3-approximation.  $\square$

We now present our algorithm ONLINE for the SBVM problem. When  $f(\cdot)$  is linearly additive, this algorithm is identical to the algorithms in [7, 17]. But in general submodular case, our algorithm based on GREEDY is simpler and clearer .

```

Input:  $G = (L, R, E)$  and function  $f(\cdot)$  on  $E$ .
Output: Matching ALG.
 $k \leftarrow \text{Binom}(|L|, \frac{1}{2})$ ;
Observe  $H \leftarrow$  the first  $k$  vertices of  $L$  and reject all edges incident to  $H$ ;
 $E' \leftarrow E \cap (H \times R)$ ;
 $M \leftarrow \text{GREEDY}(G(H, R, E'))$ ;
 $\text{ALG}, N \leftarrow \emptyset$ ;
for each subsequent  $\ell \in L \setminus H$  do
     $E_\ell \leftarrow E \cap (H \cup \{\ell\} \times R)$ ;
     $M_\ell \leftarrow \text{GREEDY}(G(H \cup \{\ell\}, R, E_\ell))$ ;
    if  $M_\ell \neq M$  then
        Let  $e_\ell$  be the edge in  $M_\ell$  incident to  $\ell$ ;
         $N = N \cup \{e_\ell\}$ ;
        if  $\text{ALG} \cup \{e_\ell\}$  is a matching then
             $\text{ALG} \leftarrow \text{ALG} \cup \{e_\ell\}$ ;
            Accept  $e_\ell$ ;
            Continue;
        end
        Reject all edges incident to  $\ell$ ;
    end
end

```

**Algorithm 6:** ONLINE for the SBVM Problem

In the algorithm, we first observe  $k = \text{Binom}(|L|, 1/2)$  number of left nodes  $H$  while rejecting all edges incident to them, where  $k$  is the binomial random variable. We build the matching  $M$  by running GREEDY on the edges  $E \cap (H \times R)$ . For any subsequent left node  $\ell$ , again we run GREEDY on the edges  $E \cap (H \cup \{\ell\} \times R)$  to build a new matching  $M_\ell$ . If there exists an edge  $e_\ell$  incident to  $\ell$  in  $M_\ell$ , we add  $e_\ell$  into  $N$ , which is a set maintained for later analysis. The edge  $e_\ell$  is accepted by our algorithm if it forms a matching with the set of accepted edges so far.

Korula and Pál [17] observed that there is a randomized offline algorithm which can simulate their online algorithm. In our case, we also describe a closely related algorithm SIMULATE as Algorithm 3. SIMULATE is easier to analyze, since it simulates the randomness of the input in ONLINE by its internal random coins. In SIMULATE, we have a reference matching  $M$ , a working set  $N$  and a result set  $S \subseteq N$ .

We start with two empty sets  $M$  and  $N$ . At each time, we pick an edge  $e = (\ell, r)$  greedily with respect to function  $f_M(\cdot)$  such that  $M \cup \{e\}$  is a matching. We then flip a coin. If the coin is head,  $e$  is added to  $M$ . Otherwise, it is added to  $N$ . In both cases, we remove all edges incident to  $\ell$ . The algorithm stops when no edge can be picked.

We introduce the two observations about SIMULATE from [17]: Once any edge incident to a vertex  $\ell \in L$  has been picked, no other edge incident to  $\ell$  will be picked later since we remove all of them. Second, multiple edges incident to  $r \in R$  might be picked until one of these edges is added

```

Input:  $G = (L, R, E)$  and function  $f(\cdot)$ 
Output: The set of selected edges  $S$ 
 $M, N, S \leftarrow \emptyset$ ;
while  $\exists e^* = (\ell^*, r^*) = \arg \max_{e \in E} \{f_M(e) \mid M \cup \{e\} \text{ is a matching}\}$  do
    Flip a coin with probability  $\frac{1}{2}$  of head;
    if head then  $M \leftarrow M \cup \{e\}$ ;
    else  $N \leftarrow N \cup \{e\}$ ;
    Remove all edges incident to  $\ell^*$  from  $E$ ;
end
foreach edge  $e = (\ell, r) \in N$  do
    Add  $e$  to  $S$  if  $e$  is the only edge incident to  $r$  in  $N$ ;
end
return  $S$ ;

```

**Algorithm 7:** SIMULATE

to  $M$ . As a consequence, at the end of SIMULATE,  $M$  is a matching, but  $N$  might not be.

Our final pruning step takes care the case that  $N$  is not a matching, i.e., there are multiple edges incident to the same vertex of  $R$  in  $N$ . The final output is a matching  $S$ . We will prove that  $S$  is a constant approximation. Before that, we justify the usefulness of SIMULATE by the following lemma, which is implicitly assumed in [17].

**Lemma C.2** *The sets of edges of  $M$  and  $N$  by SIMULATE has the same joint distribution as the  $M$  and  $N$  generated by ONLINE with a random permutation of the left nodes  $L$ .*

**Proof:** We couple the randomness in SIMULATE and ONLINE as follows. In ONLINE, the following randomness is used: (a) the random permutation of  $L$ ; (b) a random number  $k = \text{Binom}(|L|, 1/2)$ . For any particular  $k$  and a permutation of  $L$ , the probability is  $\binom{n}{k} 2^{-n} (n!)^{-1} = 2^{-n} (k!(n-k)!)^{-1}$ .

In SIMULATE, we can associate each node in  $L$  with a fair random coin. When SIMULATE accesses a coin in processing an edge incident to node  $\ell$ , we simply toss the coin associated with  $\ell$ . Let  $H$  be the set of nodes in  $L$  whose coin is head and  $T = L \setminus H$ . Define  $k = |H|$ . We append the randomness of SIMULATE by applying random permutations on  $H$  and  $T$ . A permutation of  $L$  is the concatenation of  $H$  and  $T$ . Consider any particular permutation  $p$  of  $L$  and  $k$ . Let  $H'$  be the first  $k$  nodes in the permutation. In order to generate the permutation, all the coins associated with  $H'$  must be head and all other coins must be tail. Conditioned on  $|H| = k$ , the probability that  $H' = H$  is  $\binom{n}{k}^{-1}$ .

$$\begin{aligned}
 \Pr[p \wedge |H| = k] &= \Pr[p \mid H' = H] \cdot \Pr[H' = H \mid |H| = k] \cdot \Pr[|H| = k] \\
 &= (k!(n-k)!)^{-1} \cdot \binom{n}{k}^{-1} \cdot \binom{n}{k} 2^{-n} = 2^{-n} (k!(n-k)!)^{-1}.
 \end{aligned}$$

Therefore, the probabilities of having a particular permutation and  $k$  are the same in the two algorithms. It is then sufficient to show that both algorithms generate the same  $M$  and  $N$ , given a fixed permutation of  $L$  and  $H$ .

Notice that  $M = \text{GREEDY}(H, R, E \cap (H \times R))$  in both algorithms, which must be identical. Now we prove for  $N$ . Consider an  $e = (\ell, r)$  of  $N$  in ONLINE. It is in  $M_\ell = \text{GREEDY}(G(H \cup \{\ell\}, R, E_\ell))$ . Assume  $e$  is the  $i$ -th edge added in  $M_\ell$ . Let  $M_{i-1}$  be the first  $i-1$  edges added in  $M$ . By GREEDY,  $e_\ell = \arg \max_{e \in E_\ell} \{f_{M_{i-1}}(e) \mid M_{i-1} \cup \{e\} \text{ is a matching}\}$ . In SIMULATE, after the first  $i-1$  edges are added into  $M$ ,  $e_\ell$  must be tested before the  $i$ th edge is added into  $M$ . (At least one edge incident to  $\ell$  must be tested during this period.) Since the coin associated with  $\ell$  is tail,  $e_\ell$  is added to  $N$  in SIMULATE.

On the other hand, if  $e_\ell$  is in  $N$  in SIMULATE. In the execution of ONLINE, adding  $\ell$  into  $H$  will definitely change the solution of GREEDY. In particular,  $e_\ell$  will be added into  $N$  in ONLINE, since it is the optimal one in all edges incident to  $\ell$ .  $\square$

## C.1 Analysis of ONLINE and SIMULATE

Let OPT be the optimal matching in  $G$ . We first study the performance of  $M$ . The following lemma is very intuitive.

**Lemma C.3**  $\mathbb{E}[f(M)] \geq \frac{1}{6}f(\text{OPT})$ .

**Proof:** Let OPT' denotes the optimal matching of  $H \times R$ . Notice that each edge in OPT is in  $H \times R$  with probability  $1/2$ . By submodularity,  $\mathbb{E}[f(\text{OPT}')] \geq \frac{1}{2}f(\text{OPT})$ . Recall that  $M = \text{GREEDY}(G(H, R, E \cap (H \times R)))$ . By Lemma C.1,  $M$  is a 3-approximation of OPT'. Therefore  $\mathbb{E}[f(M)] \geq \frac{1}{6}f(\text{OPT})$ .  $\square$

Let  $E_s$  be the set of edges picked in SIMULATE. Define  $M_e$  be the set of edges in  $M$  when  $e$  is picked. For each node  $r \in R$ , define  $e_r \in E_s$  as the first edge incident to  $r$  picked in SIMULATE; define  $m_r \in M$  as the edge incident to  $r$ . By the submodularity of  $f(\cdot)$  and the greedy algorithm, we have  $f_{M_{e_r}}(e_r) \geq f_{M_{m_r}}(m_r)$ . (Both  $e_r$  and  $m_r$  are random edges, which may not exist. We define the  $f_T(e) = 0$  when  $e$  does not exist.) Since  $f(M) = \sum_{r \in R} f_{M_{m_r}}(m_r)$ , the following proposition can be directly implied.

**Proposition C.4**  $\sum_{r \in R} f_{M_{e_r}}(e_r) \geq f(M)$ .

In order to show the performance of  $f(S)$ , it is sufficient to compare with  $\sum_{r \in R} f_{M_{e_r}}(e_r)$ . However, it is not even intuitively clear why there exists a relationship between these two quantities. In particular, recall that in SIMULATE, we always pick edges greedily *respect to the current  $M$* , regardless of  $N$  and  $S$ .

Instead, we inspect the function  $F(M, S) = f(S) + (f(M) + f(S) - f(M \cup S))$  during the execution of SIMULATE. The second term  $f(M) + f(S) - f(M \cup S)$  can be viewed as the intersection of  $M$  and  $S$ . Intuitively, when we pick an edge  $e$  in SIMULATE, if  $f_M(e)$  and  $f_S(e)$  are comparable to each other, the growth of  $f(S)$  is good. On the other hand, in case that  $f_S(e) \ll f_M(e)$ , with probability  $1/2$ ,  $e$  is added into  $M$ , in which case the “intersection” between  $M$  and  $S$  grows. The proof of the following lemma concretely implements this idea.

**Lemma C.5**  $\mathbb{E}[f(S)] \geq \frac{1}{8}\mathbb{E}[\sum_{r \in R} f_{M_{e_r}}(e_r)]$

**Proof:** Consider the function  $F(M, S)$ . Let  $M_r$  and  $S_r$  be the set of edges in  $M$  and  $S$  respectively when the edge  $e_r$  is picked in SIMULATE. Denote  $M'_r$  and  $S'_r$  as the set of edges in  $M$  and  $S$  after  $e_r$  is processed.



Define  $\Delta_r = F(M'_r, S'_r) - F(M_r, S_r)$ .  $F(M, S)$  is monotonically increasing when edges are processed in SIMULATE. Therefore,  $F(M, S) \geq \sum_{r \in R} \Delta_r$ . Let  $\mathcal{F}_r$  be the sub- $\sigma$ -algebra encoding all randomness up to the time  $e_r$  is picked in SIMULATE. Notice that  $M_r$ ,  $S_r$  and  $e_r$  are  $\mathcal{F}_r$  measurable.

$$\begin{aligned} \mathbb{E}[\Delta_r | \mathcal{F}_r] &= \Pr[e_r \in S | \mathcal{F}_r](2f_{S_r}(e_r) - f_{M_r \cup S_r}(e_r)) + \Pr[e_r \in M | \mathcal{F}_r](f_{M_r}(e_r) - f_{M_r \cup S_r}(e_r)) \\ &\geq \frac{1}{4}(2f_{S_r}(e_r) - f_{M_r \cup S_r}(e_r)) + \frac{1}{2}(f_{M_r}(e_r) - f_{M_r \cup S_r}(e_r)) \\ &\geq \frac{1}{2}f_{M_r}(e_r) - \frac{1}{4}f_{M_r \cup S_r}(e_r) \\ &\geq \frac{1}{4}f_{M_r}(e_r). \end{aligned}$$

The last two inequalities come from the submodularity of  $f(\cdot)$ . Therefore,

$$2\mathbb{E}[f(S)] \geq \mathbb{E}[F(M, S)] = \sum_{r \in R} \mathbb{E}[\Delta_r] = \sum_{r \in R} \mathbb{E}_{\mathcal{F}_r}[\mathbb{E}[\Delta_r | \mathcal{F}_r]] \geq \frac{1}{4} \sum_{r \in R} \mathbb{E}[f_{M_r}(e_r)]$$

□

Combing the analysis above, we have the following theorem.

**Theorem C.6** *There are 48-competitive algorithms for the SBVM problem and the submodular matroid secretary problem under a transversal matroid with monotonically increasing submodular valuation functions.*

## D Algorithms for the partition matroids

In this section, we develop constant competitive algorithms for the submodular matroid secretary problem under partition matroids. Recall that an independent set in a partition matroid contains at most one element from each group.

### D.1 Monotone submodular functions

We first discuss the case when  $f(\cdot)$  is a monotonically increasing submodular function. Our algorithm is extremely simple, though the analysis for its performance is subtle to obtain. In fact, we parallel run  $r$  *classical secretary* algorithms on each group of elements in PARTITION.

Let  $\text{Id}(\cdot)$  be the function which returns the id of the group for a particular element. For group  $i$ , we observe the first  $n_i/2$  elements and reject all of them. For each sub-sequential element in  $U_i$ , we accept it if it is better than all elements we have seen in  $U_i$ , with respect to  $f_S(\cdot)$  where  $S$  is the current set of accepted elements. (If the marginal value is negative, we do not accept the current element and any element from this group.)

Notice that  $S$  is changing in our algorithm. Consider a particular group  $i$  and current  $S$ . It could be the case that we have missed the “best element” in group  $i$  respect to  $f_S(\cdot)$  simply because  $S$  was different when it arrived. So it is rather counter-intuitive that this algorithm is a constant competitive algorithm.

We require more notations. Let  $\text{OPT} = \arg \max\{f(T) | T \in \mathcal{I}\}$  be the optimal solution and  $S \in \mathcal{I}$  be the output of our algorithm. For a given permutation  $p$  of elements in  $U$ , let  $S^p$  be the

```

Input: Partition matroid  $\mathcal{M} = (U, \mathcal{I})$ , function  $f(\cdot)$ 
Output: The set of selected elements  $S \in \mathcal{I}$ 
 $\forall i \in [r], T_i \leftarrow \emptyset, \forall j \in [n_i], b_i^j \leftarrow \text{false}; S \leftarrow \emptyset;$ 
for each element  $e \in U$  do
     $T_{\text{Id}(e)} \leftarrow T_{\text{Id}(e)} \cup \{e\};$ 
    if  $(\forall j \in [n_{\text{Id}(e)}], b_{\text{Id}(e)}^j \text{ is false}) \wedge (|T_{\text{Id}(e)}| > \frac{n_{\text{Id}(e)}}{2}) \wedge (e = \arg \max_{e' \in T_{\text{Id}(e)}} \{f_S(e')\})$  then
         $b_{\text{Id}(e)}^{|T_{\text{Id}(e)}|} \leftarrow \text{true};$ 
        if  $f_S(e) > 0$  then
             $S \leftarrow S \cup \{e\}; \text{Accept } e;$ 
        end
    end
end
return  $S;$ 

```

**Algorithm 8:** PARTITION

output of our algorithm. Denote  $\text{OPT}_i$  (resp.  $e_i, e_i^p$ ) for  $i \in [r]$  as the element of  $\text{OPT}$  (resp.  $S, S^p$ ) in group  $i$ . Let  $S_i$  (resp.  $S_i^p$ ) be the set of elements accepted by our algorithm when one of the  $b_i^j$ s is set to true in PARTITION. If all  $b_i^j$  are false after the algorithm terminates, we set  $S_i = S$ .

**Remark on Non-existence.** In general,  $e_i$  and  $e_i^p$  may not exist. We evaluate  $f_{S_i}(e_i)$  and  $f_{S_i^p}(e_i^p)$  to 0 in this case. When  $f(\cdot)$  is not monotone,  $\text{OPT}_i$  may not exist as well, in which case  $f_T(\text{OPT}_i) = 0$  for any  $T \subseteq U$ .

Our analysis crucially depend on the following definition to group the permutations of the arriving elements.

**Definition D.1 (Permutation group)** For a permutation  $p$  of elements  $U$ , define  $P_i(p)$  be the positions of  $U_i$  in  $p$ . For each  $i \in [r]$ , define the following equivalence relationship on permutations. For two permutations  $\mathbf{x} = \{x_j\}, \mathbf{y} = \{y_j\} \in \mathcal{P}$ , they are equivalent if and only if

$$P_i(\mathbf{x}) = P_i(\mathbf{y}) \text{ and } \forall j \notin P_i(\mathbf{x}), x_j = y_j.$$

Define  $\mathcal{X}_{-i}$  as the set of equivalence groups respect to  $i$ .

For permutations in the same group in  $\mathcal{X}_{-i}$ , they share the same positions for all elements except  $U_i$ . Define  $[p]_i \in \mathcal{X}_{-i}$  as the permutation group respect to  $i$  containing the permutation  $p$ . When the context is clear, we drop the subscript in  $[p]_i$  and write  $[p]$  as we always consider the group  $i$ .

For a permutation  $p$ , define the positions of  $U_i$  in  $p$  as  $\{\ell_1, \dots, \ell_{n_i}\}$  in increasing order. Now define  $T_j^p$  be the set of elements accepted by our algorithm before  $\ell_j$ th element arrives in  $p$ , if we force our algorithm to reject all elements in  $U_i$ . By definition, for any permutation  $q \in [p]$ ,  $T_j^q$  is identical to each other. Therefore, we write  $T_j^{[p]} = T_j^q$  for  $q \in [p]$ . We also define  $T^{[p]}$  be the elements our algorithm accepts with a permutation in  $[p]$ , if we force our algorithm to reject all elements in  $U_i$ .

We are interested in the probability that our algorithm accepts an element in group  $i$  as well as the quality of the element when it is accepted. The following lemma is critical in this section, which captures the interesting cases.

**Lemma D.2** *Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group, where  $x$  is a permutation. Let  $P_i(x) = \{\ell_1, \ell_2, \dots, \ell_{n_i}\}$  be the set of positions for elements in  $U_i$  in increasing order. Let  $e_i^j$  be the element accepted by our algorithm at position  $\ell_j$ . We have*

1.  $\forall j \in [n_i], \Pr[f(e_i^j) \geq \max_{e \in U_i} \{f_{T_j^{[x]}}(e)\} \mid p \in [x]] = \frac{1}{2(j-1)}$
2.  $\forall j \in [n_i], \Pr[b_i^j \text{ is true} \mid p \in [x]] = \frac{n_i}{2j(j-1)}$
3.  $\Pr[\forall j \in [n_i], b_i^j \text{ is false} \mid p \in [x]] = \frac{1}{2}$

**Proof:** Consider (1). This is the probability that we accept the *best element* in group  $i$  at position  $\ell_j$ , respect to the current set of accepted elements if the marginal gain is positive, which can be calculated as follows: a) the best element must at position  $\ell_j$ , which is with probability  $1/n_i$ . b) for each  $\ell_{j'}$ , for  $n_i/2 < j' < j$ , our algorithm does not pick the element at position  $\ell_{j'}$ , which is with probability  $(j' - 1)/j'$ . All these probabilities are independent to each other. Therefore, the probability for this case is

$$\frac{1}{n_i} \cdot \frac{j-2}{j-1} \cdot \frac{j-3}{j-2} \cdot \dots \cdot \frac{n_i/2}{n_i/2+1} = \frac{1}{2(j-1)}.$$

(2) is the probability that we accept *an element* at position  $\ell_j$  (if the marginal gain is positive), which is not necessarily the best element. We only need to change the term  $1/n_i$  to  $1/j$  in the analysis (a) above. Therefore, the probability is  $\frac{n_i}{2j(j-1)}$ .

Finally, the probability that we do not set any  $b_i^j$  to true in group  $i$  is

$$\Pr[\forall j \in [n_i], b_i^j \text{ is false} \mid p \in [x]] = 1 - \sum_{j=n_i/2+1}^{n_i} \Pr[b_i^j \text{ is true} \mid p \in [x]] = 1/2.$$

□

In the first and the second cases above, we mark  $b_i^j$  as true which implies  $S_i^p = T_j^{[x]}$  by definition. In the third case,  $S_i^p = T^{[x]}$ . The following lemma compares the performance of  $\text{OPT}_i$  with  $e_i$  respect to  $S_i^p$ . Notice that both  $S_i^p$  and  $e_i$  are random variables.

**Lemma D.3** *Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group. Assume the elements are arriving in a random permutation  $p$ . We have*

$$\mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x]] \geq \frac{1}{4} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x]].$$

**Proof:** By definition of  $S_i^p$  and Lemma D.2 (2),

$$\begin{aligned}
\mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x]] &= \sum_{j=n_i/2+1}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \Pr[b_i^j \text{ is true} \mid p \in [x]] \\
&\quad + \Pr[\forall j \in [j_i], b_i^j \text{ is false} \mid p \in [x]] \cdot f_{T^{[x]}}(\text{OPT}_i) \\
&\leq \frac{1}{2} \cdot f_{T^{[x]}}(\text{OPT}_i) + \sum_{j=n_i/2+1}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \frac{n_i}{2 \cdot j(j-1)} \tag{3}
\end{aligned}$$

Define  $e_i^j$  as the element accepted by our algorithm at position  $\ell_j$ . (The issue of non-existence is handled in the same way.) We have,

$$\begin{aligned}
\mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x]] &= \mathbb{E}\left[\sum_{j=n_i/2+1}^{n_i} f_{T_j^{[x]}}(e_i^j) \mid p \in [x]\right] \\
&\geq \sum_{j=(n_i/2+1)}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \Pr[f_{T_j^{[x]}}(e_i^j) \geq f_{T_j^{[x]}}(\text{OPT}_i) \mid p \in [x]] \\
&\geq \sum_{j=(n_i/2+1)}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \frac{1}{2(j-1)} \text{ /* Lemma D.2 (1) */} \\
&\geq \sum_{j=(n_i/2+1)}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \frac{1}{2(j-1)} \cdot \frac{n_i}{2 \cdot j} \text{ /* } n_i \leq 2j \text{ */} \\
&\geq \frac{1}{2} \left( \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x]] - (1/2) \cdot f_{T^{[x]}}(\text{OPT}_i) \right) \text{ /* Eq. 3 */} \\
&\geq \frac{1}{4} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x]].
\end{aligned}$$

The last inequality comes from the submodularity property of  $f(\cdot)$  since  $S_i^p \subseteq T^{[x]}$  for  $p \in [x]$ .  $\square$

It is then straightforward to bound the performance of  $f(S)$ .

**Lemma D.4**  $\mathbb{E}[f(S)] \geq \frac{1}{5} \mathbb{E}[f(S \cup \text{OPT})]$

**Proof:** By linearity of expectation, we have

$$\mathbb{E}[f(S)] = \mathbb{E}\left[\sum_{i \in [r]} f_{S_i}(e_i)\right] = \sum_{i \in [r]} \mathbb{E}[f_{S_i}(e_i)] = \sum_{i \in [r]} \sum_{[x] \in \mathcal{X}_{-i}} \mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x]] \Pr[p \in [x]]. \tag{4}$$

By Lemma D.3, the submodularity of  $f(\cdot)$  and the definition of  $S^p$ , we have

$$\begin{aligned}
\mathbb{E}[f(S)] &\geq \frac{1}{4} \sum_{i \in [r]} \mathbb{E}_{[x] \in \mathcal{X}_{-i}} \left[ f_{S_i^p}(\text{OPT}_i) \mid p \in [x] \right] \geq \frac{1}{4} \sum_{i \in [r]} \mathbb{E}_{[x] \in \mathcal{X}_{-i}} \left[ f_{S^p}(\text{OPT}_i) \mid p \in [x] \right] \\
&= \frac{1}{4} \sum_{i \in [r]} \mathbb{E}[f_S(\text{OPT}_i)].
\end{aligned}$$

Define  $\text{OPT}_{<i}$  as the set of elements in  $\text{OPT}$  with group id smaller than  $i$ . Continuing the previous derivation, by submodularity, we have

$$\mathbb{E}[f(S)] \geq \frac{1}{4} \sum_{i \in [r]} \mathbb{E}[f_{S \cup \text{OPT}_{<i}}(\text{OPT}_i)] = \frac{1}{4} \mathbb{E}[f_S(\text{OPT})] = \frac{1}{5} \mathbb{E}[f(S \cup \text{OPT})].$$

The last equality comes from  $f_S(\text{OPT}) = f(S \cup \text{OPT}) - f(S)$ .  $\square$

When  $f(\cdot)$  is monotonically increasing, we already have a constant competitive algorithm.

**Theorem D.5** *There is a 5-competitive algorithm for the matroid secretary problem under a partition matroid when the valuation function is a monotonically increasing submodular function.*

## D.2 Non-monotone submodular functions

We have shown that PARTITION is 5-competitive when  $f(\cdot)$  is a monotonically increasing submodular function. Notice that, all the results in the previous section can be carried over to the case with non-monotone submodular functions. In order to achieve constant competitive ratio in the non-monotone case, we need the following simple result. Notice that we always assume  $f(\cdot) \geq 0$ .

**Proposition D.6** *Let  $f(\cdot)$  be a non-negative submodular function. For any sets  $A, B, C \subseteq U$  such that  $A \cap B = \emptyset$ , we have:*

$$f(A \cup C) + f(B \cup C) \geq f(C).$$

The proof is very simple. Notice that  $(A \cup C) \cap (B \cup C) = C$  as  $A \cap B = \emptyset$ . The inequality comes from the submodularity and non-negativeness of  $f(\cdot)$ .

**Algorithm.** We first decide to run our algorithm PARTITION in mode  $A$  or  $B$  by a random choice. If we are in mode  $A$ , when we want to accept an element  $e$  in group  $i$ , we flip a coin. If the coin is head, we accept the element. Otherwise, we reject this element and never accept any element in group  $i$  in the future. Symmetrically, we do the opposite in mode  $B$ , i.e., only accept the element if the coin is tail.

Our modification simplifies the process in [14]. Denote  $\text{ALG}_A$  (resp.  $\text{ALG}_B$ ) as the output of our algorithm in mode  $A$  (resp.  $B$ ). Notice that  $\text{ALG}_A$  and  $\text{ALG}_B$  have the same distribution and  $\text{ALG}_A \cap \text{ALG}_B = \emptyset$ . So it is sufficient to show

$$\mathbb{E}[\text{ALG}_A] \geq \frac{1}{9} \mathbb{E}[\text{ALG}_A \cup \text{OPT}]. \quad (5)$$

For simplicity, we assume we are in mode  $A$ . To capture the randomness used by the random coins in our algorithm, we associate each *element* with a fair coin. Let  $Q$  be the result of all the tosses of the random coins with the elements.

Now let  $e_i^{p,Q}$  be the element our algorithm accepts in group  $i$  with a fixed  $Q$  and a permutation  $p$ . Again  $S_i^{p,Q}$  is the set of elements our algorithm accepts before  $e_i^{p,Q}$  is accepted. The non-existence issues are handled in the same way as before. The next lemma is almost identical to Lemma D.2. The only difference is when we are about to accept an element, we only accept it if the coin associated with it is head.

**Lemma D.7** Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group. Let  $P_i([x]) = \{\ell_1, \ell_2, \dots, \ell_{n_i}\}$  be the set of position for elements  $U_i$  in increasing order. Let  $Q_{-i}$  be the sub- $\sigma$ -algebra encoding all the tosses of the coins for elements other than  $U_i$ . Let  $e_i^j$  be the element accepted by our algorithm at position  $\ell_j$ . We have

1.  $\forall j \in [n_i], \Pr[b_i^j \text{ is true} \wedge f(e_i^j)] \geq \max_{e \in U_i} \{f_{T_j^{[x], Q_{-i}}}(e)\} \mid p \in [x], Q_{-i}] = \frac{1}{4(j-1)}$
2.  $\forall j \in [n_i], \Pr[b_i^j \text{ is true} \mid p \in [x], Q_{-i}] = \frac{n_i}{2j(j-1)}$
3.  $\Pr[\forall j \in [n_i], b_i^j \text{ is false} \mid p \in [x], Q_{-i}] = \frac{1}{2}$

Notice that in the following lemma  $f_{S_i^p}(\cdot)$  is a function defined on a random set  $S_i^p$  even  $p$  is fixed, i.e., it depends on  $Q$ .

**Lemma D.8** Let  $[x] \in \mathcal{X}_{-i}$  be a permutation group. Assume the elements are arriving in a random permutation  $p$  and we are in mode A. Let  $Q_{-i}$  be the sub- $\sigma$ -algebra encoding all the tosses of the coins associated with elements other than  $U_i$ . We have

$$\mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x], Q_{-i}] \geq \frac{1}{8} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}].$$

**Proof:** Let  $\{\ell_1, \ell_2, \dots, \ell_{n_i}\}$  be the set of positions in  $P_i(p)$  in increasing order for any  $p \in [x]$ . Define  $T_j^{[x], Q_{-i}}$  be the set of elements accepted in our algorithm before the  $\ell_j$ th element arrives in  $p \in [x]$ , if we force the algorithm to reject elements in  $U_i$ . Also define  $T^{[x], Q_{-i}}$  to be the set of elements our algorithm accepts if we force the algorithm to reject elements in  $U_i$ .

Therefore, by definition of  $S_i^p$  and Lemma D.7 (2),

$$\begin{aligned} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}] &= \sum_{j=n_i/2+1}^{n_i} \Pr[b_i^j \text{ is true} \mid p \in [x], Q_{-i}] \cdot f_{T_j^{[x], Q_{-i}}}(\text{OPT}_i) \\ &\quad + \Pr[\forall j \in [n_i], b_i^j \text{ is false} \mid p \in [x], Q_{-i}] \cdot f_{T^{[x], Q_{-i}}}(\text{OPT}_i) \\ &= \frac{1}{2} f_{T^{[x], Q_{-i}}}(\text{OPT}_i) + \sum_{j=n_i/2+1}^{n_i} \frac{n_i}{2j(j-1)} f_{T_j^{[x], Q_{-i}}}(\text{OPT}_i) \end{aligned} \quad (6)$$

Define  $e_i^j$  as the element accepted by our algorithm at position  $\ell_j$ . (The issue of non-existence is handled in the same way.) We have,

$$\begin{aligned}
\mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x], Q_{-i}] &= \sum_{j=n_i/2+1}^{n_i} \mathbb{E}[f_{S_i^p}(e_i^j) \mid p \in [x], Q_{-i}] \\
&= \sum_{j=n_i/2+1}^{n_i} \mathbb{E}[f_{T_j^{[x], Q_{-i}}}(e_i^j) \mid p \in [x], Q_{-i}] \\
&\geq \sum_{j=n_i/2+1}^{n_i} \Pr[f_{T_j^{[x], Q_{-i}}}(e_i^j) \geq f_{T_j^{[x], Q_{-i}}}(\text{OPT}_i) \mid p \in [x], Q_{-i}] \cdot f_{T_j^{[x], Q_{-i}}}(\text{OPT}_i) \\
&\geq \sum_{j=n_i/2+1}^{n_i} \frac{1}{4(j-1)} f_{T_j^{[x], Q_{-i}}}(\text{OPT}_i) \quad /* \text{Lemma D.7 (1)} */ \\
&\geq \sum_{j=(n_i/2+1)}^{n_i} f_{T_j^{[x]}}(\text{OPT}_i) \cdot \frac{1}{4(j-1)} \cdot \frac{n_i}{2 \cdot j} \quad /* n_i \leq 2j */ \\
&\geq \frac{1}{4} \left( \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}] - \frac{1}{2} \cdot f_{T^{[x], Q_{-i}}}(\text{OPT}_i) \right) \quad /* \text{Eq. 6} */ \\
&\geq \frac{1}{8} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}].
\end{aligned}$$

The last inequality comes from the submodularity property of  $f(\cdot)$  and the fact that  $S_i^p \subseteq T^{[x], Q_{-i}}$  for  $p \in [x]$  conditioned on  $Q_{-i}$ .  $\square$

Now we are ready to bound  $f(S)$  in mode A.

**Lemma D.9**  $\mathbb{E}[f(S)] \geq \frac{1}{9} \mathbb{E}[f(S \cup \text{OPT})]$

**Proof:** By linearity of expectation, we have

$$\mathbb{E}[f(S)] = \mathbb{E}\left[\sum_{i \in [r]} f_{S_i}(e_i)\right] = \sum_{i \in [r]} \mathbb{E}[f_{S_i}(e_i)] = \sum_{i \in [r]} \sum_{[x] \in \mathcal{X}_{-i}, Q_{-i}} \mathbb{E}[f_{S_i^p}(e_i) \mid p \in [x], Q_{-i}] \Pr[p \in [x] \wedge Q_{-i}]. \quad (7)$$

By Lemma D.8,

$$\begin{aligned}
\mathbb{E}[f(S)] &\geq \frac{1}{8} \sum_{i \in [r]} \sum_{[x] \in \mathcal{X}_{-i}, Q_{-i}} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}] \Pr[p \in [x], Q_{-i}] \\
&\geq \frac{1}{8} \sum_{i \in [r]} \sum_{[x] \in \mathcal{X}_{-i}, Q_{-i}} \mathbb{E}[f_{S_i^p}(\text{OPT}_i) \mid p \in [x], Q_{-i}] \Pr[p \in [x] \wedge Q_{-i}] \quad /* \text{by submodularity} */ \\
&= \frac{1}{8} \sum_{i \in [r]} \mathbb{E}[f_S(\text{OPT}_i)].
\end{aligned}$$

Therefore,  $\mathbb{E}[f(S)] \geq \frac{1}{9} \mathbb{E}[f(S \cup \text{OPT})]$  using the same analysis as in the proof of Lemma D.4.  $\square$   
With exactly the same argument,  $\mathbb{E}[\text{ALG}_B] \geq \frac{1}{9} \mathbb{E}[\text{ALG}_B \cup \text{OPT}]$ . Therefore, we have



$$\begin{aligned}
\mathbb{E}[f(\text{ALG}_A)] + \mathbb{E}[f(\text{ALG}_B)] &\geq \frac{1}{9}(\mathbb{E}[f(\text{ALG}_A \cup \text{OPT})] + \mathbb{E}[f(\text{ALG}_B \cup \text{OPT})]) \\
&= \frac{1}{9}\mathbb{E}[f(\text{ALG}_A \cup \text{OPT}) + f(\text{ALG}_B \cup \text{OPT})] \\
&\geq \frac{1}{9}f(\text{OPT}).
\end{aligned}$$

Notice that our algorithm will randomly choose  $\text{ALG}_A$  or  $\text{ALG}_B$  based on the random choice before any element arrives. Clearly, this is an 18-competitive algorithm.

**Theorem D.10** *There is an 18-competitive algorithm for the matroid secretary problem under a partition matroid when the valuation function is a submodular function.*

## E Conclusion

In this paper, we study constant competitive algorithms for the submodular matroid secretary problem under transversal matroids and partition matroids. Some of our results can be extended to non-monotone submodular functions as well as some online matching problems. There are still a lot of problems remain open.

One interesting problem is to design constant competitive algorithms for *non-monotone* submodular functions under transversal matroids. In particular, our algorithm is using the greedy algorithm as a subroutine. When the submodular function is monotone, the greedy algorithm guarantees a constant approximation under matroid constraints. This property however fails when the function is non-monotone.

Comparing with the original matroid secretary problem, we are still not able to address the laminar matroids and the general truncated matroids in the context of submodular functions. One reason is the techniques used in cases with laminar matroids [15] and truncated matroids [3] are heavily relied on the distribution of the *optimal* solutions. In case of submodular functions, it is clear that one has to relax our attention to some kind of greedy solutions.

Ultimately, the conjecture on constant competitive algorithms for the matroid secretary problem under general matroids is still widely open even with linearly additive functions. However, we suspect that the submodular case might exhibit constant competitive algorithms as well, at least for monotone functions. On another direction, one might be able to provide constant competitive algorithms for the general submodular matroid secretary problem in random assignment model, extending the results in [18].